Non-conforming Galerkin finite element method for symmetric local absorbing boundary conditions

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Abstract

We propose a new solution methodology to incorporate symmetric local absorbing boundary conditions involving higher tangential derivatives into a finite element method for solving the 2D Helmholtz equations. The main feature of the method is that it does not requires the introduction of auxiliary variable nor the use of basis functions of higher regularity on the artificial boundary. The originality lies in the combination of C^0 continuous finite element spaces for the discretization of second order operators with discontinuous Galerkin-like bilinear forms for the discretization of differential operators of order four and above. The method proves to limit the computational costs than methods based on auxiliary variables as soon as the order of the absorbing boundary condition is greater than three or the order of the numerical scheme is greater than two. The article includes the numerical analysis of the discrete discontinuous Galerkin variational formulation. Numerical results show that the method does not hamper the order of convergence of the finite element method, if the polynomial degree on the boundary is sufficiently high.

Keywords: Interior Penalty Galerkin finite element method, Local absorbing boundary condition

Contents

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1. Introduction

We consider a second order partial differential equations in the connected Lipschitz domain $\Omega \in \mathbb{R}^d$, $d = 2, 3$ with *symmetric local absorbing boundary condition* (see [24, Eq. 3.14] and [46]), which reads for $d = 2$

$$\partial_\nu u + \sum_{j=0}^{J} (-1)^j \partial_\Gamma^j (\alpha_j \partial_\Gamma^j u) = g, \tag{1}$$

on a closed subset Γ of the boundary $\partial\Omega$ for smooth enough function α_j on Γ. Here, ∂_ν and ∂_Γ denote the normal and tangential (surface) derivatives on Γ, $\partial_\nu := \nu \cdot \nabla$, $\partial_\Gamma = \tau \cdot \nabla$, where n is the outer normalised normal vector on Γ and τ the normalised tangential vector. Then, $J \in \mathbb{N}_0 \cup \{-1\}$ is the order of highest (second) derivatives, and the condition has given names for special cases of J. For $J = -1$ it is called as *Neumann*, for $J = 0$ as *Robin* and for $J = 1$ as *Wentzell boundary condition* [3, 9, 15, 17, 18, 55, 56]. The symmetric local absorbing boundary conditions in three dimensions take a similar form, see Chap. 3.1. Here, we also discuss a special class of non-symmetric local absorbing boundary conditions.

Local absorbing boundary conditions (1) are stated for example on artifical boundaries of truncated, originally infinite domains, to approximate radiation or decay conditions. Then, the functions α_j correspond to the partial differential equation outside Ω and a better approximation is obtained by pushing the artifical boundary further to infinity or by adding further terms, , increasing J. An overview over those conditions, which are in the context of wave propagation problems also called transparent or non-reflecting boundary condition, see [20, 30].

If a possibly bounded subdomain correspond to a highly conducting body in electromagnetics the fields can be computed approximately by a formulation in the exterior of the conductor with so called surface or generalised impedance boundary conditions on the conductor surface [26, 33, 42, 51, 52, 60]. Similar impedance boundary conditions have been derived for thin dielectric coatings on perfect conducting bodies [4, 8, 16, 40] or for viscosity boundary layers in acoustics [48]. For thin layers inside the domain impedance transmission conditions as maps between Dirichlet and Neumann jump and mean are derived [34, 36, 39, 45, 49], even for microstructured layers [14] for which Γ is usually taken as mean-line.

The derivation of these local ABCs is often by asymptotic expansion techniques or a truncation of Fourier series, where at least for the rigorous error estimates the boundary Γ and the local structure, hence, the functions α_j are assumed to be smooth. The local ABCs may be applied in each smooth part for piecewise smooth boundaries Γ, *e. g.*, domains with corners, or for piecewise smooth functions α_j which may have jumps. In this case the higher surface derivatives $\partial_\Gamma^j \alpha_j \partial_\Gamma^j$ are not necessary weak derivatives on the whole boundary Γ and corner conditions [53] have to supplement the ABC. To our knowledge these conditions have not been mathematically analysed so far and we restrict ourselves to $\Gamma \in C^\infty$ with ring topology and $\alpha_j \in C^\infty$.

With these assumptions the weak form of (1), after j-time integration by parts of the j-th term along Γ is given by

$$\int_\Gamma \partial_\nu u v + \sum_{j=0}^{J} \alpha_j \partial_\Gamma^j u \, \partial_\Gamma^j v \, \mathrm{d}\sigma(x) = \int_\Gamma g v \, \mathrm{d}\sigma(x). \tag{2}$$

It includes only surface integrals and no boundary terms appear on points of lower smoothness. If the local ABCs are taken for the Helmholtz equation with homogeneous Neumann boundary conditions on $\partial\Omega \backslash \Gamma$ we can write a variational formulation as: Seek $u_J \in V_J := H^1(\Omega) \cap H^J(\Gamma)$ such that

$$\mathsf{a}_J(u_J, v) := \int_\Omega \left(\nabla u_J \cdot \nabla \overline{v} - \kappa^2 u_J \overline{v}\right) \mathrm{d}x + \sum_{j=0}^{J} \int_\Gamma \alpha_j \partial_\Gamma^j u_J \, \partial_\Gamma^j \overline{v} \, \mathrm{d}\sigma(x) = \langle f_J, v \rangle \quad \forall\, v \in V_J, \tag{3}$$

where f_J corresponds to the source terms.

If only second derivatives are present, *i.e.*, for the Neumann, Robin and Wenttzel conditions, a numerical realisation with usual piecewise continuous finite element methods is straightforward. For $J \geq 2$, the usual finite element spaces are not any more contained in the natural space V_J of the continuous formulation. Even so the use of trial and test functions with $C^{(J-1)}$-continuity (at least) along Γ [23, 24] or auxiliary unknowns [21, 22] have been proposed local ABCs with higher-derivatives than two have rarely been used.

In this article we propose as an alternative nonconforming interior penalty finite element method for the usual continuous finite element spaces of at least order $J-1$, in which for each $j \geq 2$ in (1) additional terms on the nodes of the boundary Γ appear (Section 2). For fourth order PDEs a similar approach has been introduced by Brenner and Sung [12]. Some local ABCs of higher order do not take the symmetric form (1) and incorporate surface derivatives of odd orders. For this case we derive the additional terms in Sect. 2.5. Symmetric local ABCs and a class of non-symmetric local ABCs in three dimensions and the related interior penalty formulation will given in Section 3. The numerical analysis of the numerical method proposed in the article will be presented for the case of local symmetric ABCs in two dimension in Section 4. It relies on analytical results of such conditions given in [46]. The theoretical convergence results of Section 5 are validated by a series of numerical experiments in Section 5.

2. Interior penalty finite element formulation in 2D

For the derivation of the interior penalty formulation we need the following regularity result.

Lemma 2.1. *Let $u_J \in V_J$ be solution of (2) with $\inf_{x\in\Gamma} |\alpha_J| > 0$. Then, $u_J \in C^\infty(\Gamma)$.*

Proof. The proof is a simple generalization of the proof of Lemma 2.8 in [46] from constants α_j to $\alpha_j \in C^\infty$. $\square$

2.1. Definition of the C^0-continuous finite element spaces

The presented non-conforming finite element method is based on a mesh $\mathcal{M}_h$ of the computation domain Ω (see Fig. 1) consisting of possibly curved triangles $\mathcal{T}_h$ and curved quadrilaterals $\mathcal{Q}_h$, which are disjoint and which fill the computational domain, , $\Omega = \bigcup_{K\in\mathcal{M}_h} \overline{K}$. Each cell K in $\mathcal{T}_h$ or $\mathcal{Q}_h$ can be represented through a smooth mapping F_K from a single reference triangle $\widehat{K}$ or a single reference quadrilateral $\widehat{K}$, respectively. We denote $\mathcal{E}(\mathcal{M}_h, \Gamma)$ the set of edges of $\mathcal{M}_h$ on Γ, $\mathcal{N}(\mathcal{M}_h, \Gamma)$ is the set of nodes of $\mathcal{M}_h$ on Γ and $\mathcal{N}(e)$ is the set composed of the two nodes of the external edge e. Furthermore, we define the union of all outer boundary edges and the union of all cells as

$$\Gamma_h := \bigcup_{e\in\mathcal{E}(\mathcal{M}_h,\Gamma)} e = \Gamma \setminus \bigcup_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)} n, \qquad \Omega_h := \bigcup_{K\in\mathcal{M}_h} K = \Omega \setminus \bigcup_{e\in\mathcal{E}(\mathcal{M}_h)} \overline{e}.$$

The edges $\mathcal{E}(\mathcal{M}_h, \Gamma)$ of $\mathcal{M}_h$ are possibly curved, and for the analysis we assume that they resolve exactly Γ, ,

$$\Gamma = \bigcup_{e\in\mathcal{E}(\mathcal{M}_h,\Gamma)} \overline{e}.$$

Furthermore, we assume that each edge has counter-clockwise orientation and can be represented by a smooth mapping F_e from the reference interval $(0, 1)$. The mesh width h is the largest outer diameter of the cells

$$h := \max_{K\in\mathcal{M}_h} \operatorname{diam}(K).$$

We are going to define the discretisation space. First, we denote $\widehat{K}$ a reference quadrilateral or triangle, respectively, and $\widehat{e}$ denotes either one edge of $\widehat{K}$ or the reference interval. Furthermore, we denote $\mathbb{P}_p(\widehat{K})$ the space of polynomials of maximal total degree p for the reference triangle $\widehat{K}$ and of maximal degree p in each coordinate direction for the reference quadrilateral $\widehat{K}$. The space $\mathbb{P}_p(\widehat{K})$ can be decomposed into interior bubbles, edges bubbles to one of the edges and the nodal functions. The space of interior bubbles for the reference triangle is $\mathbb{P}_p(\widehat{K}, 0) := \{\widehat{v} \in \mathbb{P}_p(\widehat{K}) : \widehat{v}|_{\partial\widehat{K}} = 0\}$ and the one of the edge bubbles related to an edge $\widehat{e}$ in $\widehat{K}$ is given by $\mathbb{P}_p(\widehat{K}, \widehat{e}) := \{\widehat{v} \in \mathbb{P}_p(\widehat{K}) : \widehat{v}|_{\partial\widehat{K}\setminus\widehat{e}} = 0, \langle\widehat{v}, \widehat{w}\rangle_{L^2(\widehat{K})} = 0 \; \forall \widehat{w} \in \mathbb{P}_P(\widehat{K}, 0)\}$.

Now, let p be a function assigning each cell $K \in \mathcal{M}_h$ and each edge $e \in \mathcal{E}(\mathcal{M}_h, \Gamma)$ a polynomial order $\mathsf{p}(K)$ or $\mathsf{p}(e)$, respectively, which are all positive integers and $\mathsf{p}(e) \geq \mathsf{p}(K)$ if $e \subset \overline{K}$. We denote $p := \min_{K \in \mathcal{M}_h} \mathsf{p}(K) \geq 1$ and $p_\Gamma := \min_{e \in \mathcal{E}(\mathcal{M}_h,\Gamma)} \mathsf{p}(e) \geq p$. To define the local solution space we denote by $\mathcal{M}_h(K)$ the set of those neighbouring cells K in $\mathcal{M}_h \backslash \{K\}$ which have a common edge with K and by $\mathcal{E}(K, \Gamma)$ the edges of K which are on the domain boundary Γ. We take as local polynomial space in K the space of polynomials with maximal degree $\mathsf{p}(K)$ on the reference element (in each direction for a quadrilateral) and with possibly additional edge bubbles related to neighbouring elements and edge bubbles related to boundary edges,

$$\mathbb{P}_{\mathsf{p}}(K) := \Big\{ v_h \in C^\infty(K) : v_h|_K \circ F_K \in \mathbb{P}_{\mathsf{p}(K)}(F_K^{-1}K) \cup \bigcup_{K' \in \mathcal{M}_h(K)} \mathbb{P}_{\min(\mathsf{p}(K),\mathsf{p}(K'))}(F_K^{-1}K, F_K^{-1}(\overline{K} \cap \overline{K}')) \cup \bigcup_{e \in \mathcal{E}(K,\Gamma)} \mathbb{P}_{\mathsf{p}(e)}(F_K^{-1}K, F_K^{-1}e) \Big\}.$$

Note, that the respective space on the reference element $\widehat{K}$ is a subset of $\mathbb{P}_{\mathsf{p}^\star(K)}(\widehat{K})$ for some $\mathsf{p}^\star(K) \geq \mathsf{p}(K)$. Here, $\mathsf{p}^\star(K)$ is the polynomial degree to represent the basis functions on K. It is larger than $\mathsf{p}(K)$ if $\mathsf{p}(K')$ is larger for one of the neighbouring elements $K' \in \mathcal{M}_h(K)$ or if a edge bubble function related to a higher polynomial order on a boundary edge has to be represented.

Then, we define the space of piecewise polynomial, continuous basis functions with polynomial orders p as

$$V_h := S^{\mathsf{p},1}(\Omega, \mathcal{M}_h) := \{ v_h \in C^0(\overline{\Omega}) : \ v_h|_K \in \mathbb{P}_{\mathsf{p}}(K) \ \}.$$

For the definition of the surface derivative, we fix the tangential vector τ to be always the counter-clockwise (or clockwise) one on whole Γ (see Fig. 1).

Note, that $V_h \subset V_{j,h} := H^1(\Omega) \cap H^1(\Gamma) \cap H^j(\Gamma_h)$ for any $j \in \mathbb{N}$. The spaces $V_{j,h}$ are equipped with the so called broken norms

$$\|v\|^2_{V_{j,h}} := \|v\|^2_{H^1(\Omega)} + \|v\|^2_{H^j(\Gamma_h)}.$$

Finally, we have to define additional notations specific to the non-conforming formulation. For each node $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$, we denote by e_n^+ and e_n^- the two external edges sharing n, such that e_n^+ follows e_n^- when going counter-clockwise (see Fig. 1). We define by $v_n^+ = \lim_{s \to 0} v(F_{e_n^+}(s))$ and by $v_n^- = \lim_{s \to 1} v(F_{e_n^-}(s))$. The jump and the mean of v on n are respectively defined by

$$[v]_n = v_n^- - v_n^+ \quad \text{ and } \quad \{v\}_n = \frac{v_n^+ + v_n^-}{2}.$$

We also denote by h_e the length of the edge e and by $h_n = \min\left(h_{e_n^+}, h_{e_n^-}\right)$.

2.2. Derivation of the interior penalty Galerkin variational formulation

Multiplying $\partial_\Gamma^j \alpha_j \partial_\Gamma^j u$ for a function $u \in C^\infty$ by $\overline{v}_h \in V_h$, which is in $C^\infty(\overline{e})$ in each edge on Γ, and applying j times integration by part in each edge e, we obtain

$$\begin{aligned}
(-1)^j \int_\Gamma \partial_\Gamma^j \alpha_j \partial_\Gamma^j u \overline{v}_h \,\mathrm{d}\sigma(x) &= (-1)^j \sum_{e \in \mathcal{E}(\mathcal{M}_h,\Gamma)} \int_e \partial_\Gamma^j \alpha_j \partial_\Gamma^j u \overline{v}_h \,\mathrm{d}\sigma(x) \\
&= \sum_{e \in \mathcal{E}(\mathcal{M}_h,\Gamma)} \int_e \alpha_j \partial_\Gamma^j u \partial_\Gamma^j \overline{v}_h \,\mathrm{d}\sigma(x) + \sum_{i=0}^{j-1} (-1)^{i+j} \sum_{n \in \mathcal{N}(\mathcal{M}_h,\Gamma)} [\partial_\Gamma^{j-i-1} \alpha_j \partial_\Gamma^j u \, \partial_\Gamma^i \overline{v}_h]_n \\
&= \int_\Gamma \alpha_j \partial_\Gamma^j u \partial_\Gamma^j \overline{v}_h \,\mathrm{d}\sigma(x) + \sum_{i=1}^{j-1} (-1)^{i+j} \sum_{n \in \mathcal{N}(\mathcal{M}_h,\Gamma)} \{\partial_\Gamma^{j-i-1} \alpha_j \partial_\Gamma^j u\}_n [\partial_\Gamma^i \overline{v}_h]_n. \qquad (4)
\end{aligned}$$

Here, we used the equivalence $[ab]_n = [a]_n \{b\}_n + \{a\}_n [b]_n$, the fact that with $u_j, \alpha_j \in C^\infty$ all jumps $\left[\partial_\Gamma^{j-i-1} \alpha_j \partial_\Gamma^j u\right]_n$, $i < j$ are zero and that with $v_h \in C^0(\Gamma)$ all jumps $\left[\overline{v}\right]_n$ are zero.

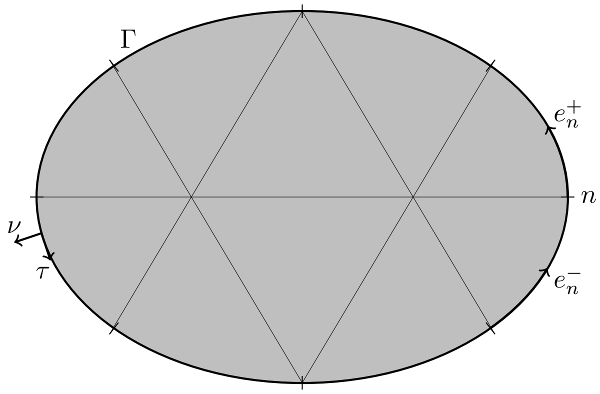

Figure 1. The triangulation $\mathcal{M}_h$ of the domain Ω with boundary Γ (unit normal vector ν and tangential vector τ are indicated) with partially curved cells. For functions on the boundary Γ the jump and average on boundary nodes n is defined using the neighbouring edges e_n^+ and e_n^- with counter-clockwise orientation.

If we are interested in symmetric bilinear forms to obtain the symmetric interior penalty Galerkin formulation (SIPG) [59], we add the terms $s\,[\partial_\Gamma^i u]_n\{\partial_\Gamma^{j-i-1}\alpha_j\partial_\Gamma^j\overline{v}_h\}_n$ with $s = 1$. We do not loose consistency as with the assumption of $u \in C^\infty(\Gamma)$ the terms $\left[\partial_\Gamma^i u\right]_n$ are in fact zero. We then obtain

$$(-1)^j \int_\Gamma \partial_\Gamma^j \alpha_j \partial_\Gamma^j u \overline{v}_h \,\mathrm{d}\sigma(x) = \int_\Gamma \alpha_j \partial_\Gamma^j u \partial_\Gamma^j \overline{v}_h \,\mathrm{d}\sigma(x) \\ + \sum_{i=1}^{j-1}(-1)^{i+j} \sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)} \Big(\{\partial_\Gamma^{j-i-1}\alpha_j\partial_\Gamma u\}_n\,[\partial_\Gamma^i\overline{v}_h]_n + s[\partial_\Gamma^i u]_n\{\partial_\Gamma^{j-i-1}\alpha_j\partial_\Gamma\overline{v}_h\}_n\Big). \tag{5}$$

Note, that alternatively $s = -1$ for the non-symmetric (NIPG) [41] or $s = 0$ for the incomplete interior penalty Galerkin formulation (IIPG) [54] can be chosen.

Finally, to ensure the coercivity of the bilinear forms (with a mesh-independent constant), we add for $j > 0$ the terms

$$\frac{\beta_j}{h_n^{2(J-j)+1}}[\partial_\Gamma^{j-1}u]_n[\partial_\Gamma^{j-1}\overline{v}_h]_n,$$

which also do not harm the consistency since $[\partial_\Gamma^{j-1}u]_n = 0$, $j = 1,\dots,J-1$.

Remark 2.2. *The assumption $u \in C^\infty(\Gamma)$ in the derivation can be lowered. In fact $u \in H^J(\Gamma)$, $\alpha_0 u \in L^2(\Gamma)$, $\alpha_j\partial_\Gamma^j u \in C^{j-1}(\Gamma)\cap L^2(\Gamma)$, $j = 1,\dots,J$ is enough for consistency. This requires $\alpha_j \in L^\infty(\Gamma)$, $j = 0,\dots,J$ and with $\partial_\Gamma^j u \in C^{j-1}(\Gamma)$ for $2j \le J$ that $\alpha_j \in C^{j-1}(\Gamma)$, $j = 1,\dots,\lfloor\frac{J}{2}\rfloor$. With these assumptions it is indeed enough to require Γ to be Lipschitz and $C^{J,1}$ in a finite partition of the boundary. However, if u is solution of the above system it is unlikely to fulfill the regularity assumptions in this case [32].*

Now, we are in the position to state the interior penalty Galerkin formulation: Seek $u_{J,h} \in V_h$ such that

$$\mathsf{a}_{J,h}(u_{J,h}, v_h) = \langle f_J, v_h\rangle\,, \quad \forall v_h \in V_h, \tag{6}$$

where

$$\mathsf{a}_{J,h}(u_h, v_h) := \int_\Omega \left(\nabla u_h\cdot\nabla\overline{v} - \kappa^2 u_h\overline{v}_h\right)\,\mathrm{d}x + \sum_{j=0}^{J}\left(\mathsf{c}_j(u_h,v_h;\alpha_j) + \mathsf{b}_{j,h;J}(u_h,v_h;\alpha_j)\right)$$

$$\mathsf{c}_j(u_h, v_h;\alpha_j) := \int_{\Gamma_h}\alpha_j\partial_\Gamma^j u_h\,\partial_\Gamma^j\overline{v}_h\,\mathrm{d}\sigma(x)\ ,$$

$\mathsf{b}_{0,h;J} = \mathsf{b}_{1,h;J} = 0$ and for $j > 1$

$$\mathsf{b}_{j,h;J}(u_h, v_h; \alpha_j) := \sum_{i=1}^{j-1} (-1)^{i+j} \sum_{n \in \mathcal{N}(\mathcal{M}_h,\Gamma)} \Big(\big\{ \partial_\Gamma^{j-i-1} \alpha_j \partial_\Gamma^j u_h \big\}_n \big[\partial_\Gamma^i \overline{v}_h \big]_n + s \, \big[\partial_\Gamma^i u_h \big]_n \big\{ \partial_\Gamma^{j-i-1} \alpha_j \partial_\Gamma^j \overline{v}_h \big\}_n \Big)$$
$$+ \sum_{n \in \mathcal{N}(\mathcal{M}_h,\Gamma)} \frac{\beta_j}{h_n^{2(J-j)+1}} [\partial_\Gamma^{j-1} u_h]_n [\partial_\Gamma^{j-1} \overline{v}_h]_n \, ,$$

where s corresponds to the symmetric, non-symmetric or incomplete interior penalty method.

2.3. Well-posedness and estimation of the discretisation error

If the finite element space is rich enough the interior penalty formulation is well-posed and we can state a result on the discretisation error. The proofs of these results will be given in Chap. 4.

Theorem 2.3. *Let* $\inf_{x\in\Gamma} \operatorname{Re}(\alpha_J) > 0$ *or* $\inf_{x\in\Gamma} |\operatorname{Im}(\alpha_J)| > 0$ *and let zero be the only solution of* (3) *with* $f_J = 0$*. Then, there exists constants* $h_{unique}, p_{unique} > 0$ *such that for all* $h < h_{unique}$ *and* $p \geq p_{unique}$ *the discrete interior penalty Galerkin variational formulation* (6) *for* $s \in [-1, 1]$ *and* β_j*,* $j = 2, \ldots, J$ *large enough admits a unique solution* $u_{J,h} \in V_h$ *and there exists a constant* $C_{J,h} > 0$ *such that*

$$\|u_{J,h}\|_{V_{J,h}} \leq C_{J,h} \|f_J\|_{V'_{J,h}} \tag{7}$$

and

$$\|u_{J,h} - u_J\|_{V_{J,h}} \leq C_{J,h} \inf_{v_h \in V_h} \|v_h - u_J\|_{V_{J,h}}. \tag{8}$$

Lemma 2.4. *Let* $J > 0$*, let the assumption of Theorem 2.3 be satisfied and let* $h < h_{unique}$*,* $p \geq p_{unique}$ *and* $p_\Gamma \geq J$*. Then, there exists a constant* $C_{J,h} > 0$ *such that for the solution* $u_{J,h} \in V_h$ *of* (6) *it holds*

$$\|u_{J,h} - u_J\|_{V_{J,h}} \leq C_{J,h} \left(\inf_{v_h \in V_h} \|v_h - u_J\|_{H^1(\Omega)} + h^{p_\Gamma - J + 1} \|f_J\|_{V'_J} \right). \tag{9}$$

The first term in the right hand side of (9) is the H^1-best-approximation error in the computational domain, which depends on one side of $\kappa(x)$ and so the regularity of u_J and on the other side on the mesh and the polynomial degree distribution p, see *e. g.* [50] for p- and hp-finite element methods. The second term is due to the discretisation of the surface differential operators in the symmetric local absorbing boundary conditions. In order to achieve a convergent discretisation the minimum p_Γ of the polynomial degrees on Γ_h has to be chosen to be at least J. For simple refinement of uniform meshes $\mathcal{M}_h$ (h-refinement) and polynomial degrees of at least p in the cells of $\mathcal{M}_h$ the polynomial degrees on the edges of $\mathcal{E}(\mathcal{M}_h, \Gamma)$ has to be chosen to be at least $p_\Gamma \geq p + J - 1$ such that the error due to the discretisation of the absorbing boundary condition does not dominate asymptotically for $h \to 0$.

2.4. Analysis of the computational costs

Our methodology requires $p + J - 1$ additional degrees of freedom per edge in $\mathcal{E}(\mathcal{M}_h, \Gamma)$ while classical methodology requires the introduction of $J - 1$ auxiliary unknowns and thus of $p(J - 1)$ additional degrees of freedom per edge in $\mathcal{E}(\mathcal{M}_h, \Gamma)$. Note that, when considering odd order ABC, the methodology proposed by Hagstrom *et.al.* [27, 28] reduces this cost to $(J - 1)/2$ auxiliary unknowns and $p(J - 1)/2$ degrees of freedom. Hence, our strategy is more costly when both p and J are small but less costly when p or $J - 1$ are greater than four. The higher p or $J - 1$ are, the more beneficial the proposed solution methodology is.

2.5. Interior penalty formulation for terms with odd tangential derivatives

The proposed interior penalty formulation can be extended to the local boundary condition involving terms with odd tangential derivatives of order $2J-1$ and less. We will derive the additional term in the variational formulation on the example of the term $\partial_\Gamma^2(\gamma\partial_\Gamma u)$ for $\gamma \in C^\infty(\Gamma)$. In analogy to terms with even derivatives we integrate by parts on Γ_h and use the fact that $\left[u\right]_n = 0$ on all $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$ to obtain

$$\int_{\Gamma_h} \partial_\Gamma^2(\gamma\partial_\Gamma u)\overline{v}\,\mathrm{d}\sigma(x) = -\int_{\Gamma_h} \partial_\Gamma(\gamma\partial_\Gamma u)\partial_\Gamma\overline{v}\,\mathrm{d}\sigma(x).$$

Then, we divide the expression into two parts and apply integration by parts on one part, and using that $\left[\partial_\Gamma u\right]_n = 0$ on all $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$

$$\int_{\Gamma_h} \partial_\Gamma^2(\gamma\partial_\Gamma u)\overline{v}\,\mathrm{d}\sigma(x) = \frac{1}{2}\int_{\Gamma_h} -\partial_\Gamma(\gamma\partial_\Gamma u)\partial_\Gamma\overline{v} + \gamma\partial_\Gamma u\partial_\Gamma^2\overline{v}\,\mathrm{d}\sigma(x) + \frac{1}{2}\sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)} \gamma_n\left\{\partial_\Gamma u\right\}_n\left[\partial_\Gamma\overline{v}\right]_n$$

where γ_n are the function values of γ on $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$. Now, adding the terms $s\gamma_n\left[\partial_\Gamma u\right]_n\left\{\partial_\Gamma\overline{v}\right\}_n$ related to the different variants of interior penalty formulations and using the identity $\gamma\partial_\Gamma u\partial_\Gamma^2\overline{v} = \partial_\Gamma u\partial_\Gamma(\gamma\partial_\Gamma\overline{v}) - \partial_\Gamma\gamma\partial_\Gamma u\partial_\Gamma\overline{v}$ we find

$$\begin{aligned}\int_{\Gamma_h} \partial_\Gamma^2(\gamma\partial_\Gamma u)\overline{v}\,\mathrm{d}\sigma(x) &= \frac{1}{2}\int_{\Gamma_h} \partial_\Gamma u\partial_\Gamma(\gamma\partial_\Gamma\overline{v}) - \partial_\Gamma(\gamma\partial_\Gamma u)\partial_\Gamma\overline{v}\,\mathrm{d}\sigma(x) - \frac{1}{2}\int_{\Gamma_h}(\partial_\Gamma\gamma)\partial_\Gamma u\partial_\Gamma\overline{v}\,\mathrm{d}\sigma(x)\\ &\quad + \frac{1}{2}\sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)} \gamma_n\left(\left\{\partial_\Gamma u\right\}_n\left[\partial_\Gamma\overline{v}\right]_n + s\left[\partial_\Gamma u\right]_n\left\{\partial_\Gamma\overline{v}\right\}_n\right).\end{aligned}$$

We observe that the formulation with those additional terms related to odd derivatives looses symmetry even for $s = 1$. Independent of the choice of s there is no need to add any further penalty term to ensure coercivity.

3. Interior penalty finite element formulation in 3D

3.1. Local absorbing boundary conditions in 3D

In three dimensions we can decompose the gradient ∇u of a function u in the contribution normal to the boundary Γ, that is $\nu\partial_\nu u = \nu(\nabla u\cdot\nu)$, and the tangential gradient $\nabla_\Gamma u := \nabla u - \nu\partial_\nu u$. Similarly, the Laplacian Δu can be decomposed into the second normal derivative $\partial_\nu^2 u = \nu^\top H(u)\nu$, where H is the Hessian matrix with all partial second derivatives, and the Laplace-Beltrami operator $\Delta_\Gamma u := \Delta u - \partial_\nu^2 u$. Note, that the latter is also given in terms of the surfacic divergence div_Γ by $\Delta_\Gamma := \mathrm{div}_\Gamma \nabla_\Gamma$ (see [37, Sect. 2.5.6] for smooth surfaces).

With the Laplace-Beltrami operator we can define the BGT condition of order [7], which is a non-reflecting boundary condition for the time-harmonic Helmholtz equation in 3D. With wave-number k it is given as the Wentzell condition

$$\partial_\nu u = \frac{1}{2}\left(-\mathrm{i}k + \frac{2}{R}\right)^{-1}\left(\Delta_\Gamma + \frac{2}{R^2} + \frac{4\mathrm{i}k}{R} + 2k^2\right)u =: \mathcal{B}_2 u, \tag{10}$$

and set on spherical boundary Γ of radius R. If the shape of the boundary is arbitrary, but smooth, a similar condition has been derived in [2], which includes a term of the form $\mathrm{div}_\Gamma(I - \frac{\mathrm{i}}{k}\mathcal{R})\nabla_\Gamma)$, where I is the identity and $\mathcal{R}$ the curvature tensor. In the framework of Givoli and Keller non-reflecting boundary conditions for ellipsoidal boundaries, taking the form of a Wentzell condition, have been proposed in [6]. Generalised impedance boundary conditions for highly conducting bodies of order 3 [26] can be found in the form of a Wentzel condition.

These conditions can be used directly with C^0-continuous finite elements with an additional bilinear form

$$\int_\Gamma \alpha uv + (\beta\nabla_\Gamma u)\cdot\nabla_\Gamma v\mathrm{d}S(x),$$

order	α_0	α_1	α_2	β_0	β_1
$\mathcal{B}_1$	$-\left(-\mathrm{i}k+\frac{1}{R}\right)$	–	–	1	–
$\mathcal{B}_2$	$2\left(-\mathrm{i}k+\frac{1}{R}\right)^2$	1	–	$-2\left(-\mathrm{i}k+\frac{2}{R}\right)$	–
$\mathcal{B}_3$	$2\left(-2\mathrm{i}k^3+\frac{9k^2}{R}+\frac{9\mathrm{i}k}{R^2}-\frac{3}{R^3}\right)$	$-3\left(-\mathrm{i}k+\frac{1}{R}\right)$	–	$4\left(-\mathrm{i}k+\frac{3}{R}\right)\left(-\mathrm{i}k+\frac{3}{2R}\right)$	1
$\mathcal{B}_4$	$8\left(k^4+\frac{8\mathrm{i}k^3}{R}-\frac{18k^2}{R^2}-\frac{12\mathrm{i}k}{R^3}+\frac{3}{R^4}\right)$	$8\left(-\mathrm{i}k+\frac{3}{R}\right)^2$	1	$-8\left(-\mathrm{i}k+\frac{2}{R}\right)\left(-k^2-\frac{6\mathrm{i}k}{R}+\frac{6}{R^2}\right)$	$-4\left(-\mathrm{i}k+\frac{1}{R}\right)$

Table 1. Coefficients of the BGT conditions.

only, where some scalar function α and some possibly tensorial function β appears.

Patlashenko and Givoli [38] introduce symmetric local absorbing boundary conditions of any order J in three dimensions

$$\partial_\nu u = \sum_{j=0}^{J} \alpha_j \,(-1)^j \Delta_\Gamma^j u, \tag{11}$$

where α_j are scalar constants. In this case (11) can be seen as a generalisation of (1) in three dimensions. Harari has derived in the framework of [24] parameters α_j for non-reflecting boundary conditions for the exterior of a sphere [29].

To our best knowledge, the usage of local absorbing boundary conditions with higher tangential derivatives than two in a finite element context has only been reported with basis functions with higher regularity on Γ, but not with the usual C^0-continuous basis functions only.

In the following we would like to discuss a more general case, where the Laplace-Beltrami operator is applied once or more to the normal derivative $\partial_n u$, and the highest derivative to u and $\partial_n u$ have same order. These conditions have the form

$$0 = \sum_{j=0}^{J} (-1)^j \left(\alpha_j \,\Delta_\Gamma^j u + \beta_j \,\Delta_\Gamma^j \partial_n u\right), \tag{12}$$

where $\beta_0 \neq 0$. Hence, we can assume without loss of generality that $\beta_0 = -1$. Those conditions arise in the derivation of robust impedance conditions from a Padé approximation [26], but also the BGT conditions [7] of odd order take this form as illustrated in Tab. 1. Note, that the BGT conditions of order 1 and 2 can be written as (11).

To derive weak formulation for (11) or (12) in general we introduce by misuse of notation $\Delta_\Gamma^{1/2} := \nabla u$, which is a vector valued function, and define in addition to Δ_Γ^j for integer j the vector $\Delta_\Gamma^j := \nabla \Delta_\Gamma^{j-1/2}$ if $2j$ is an integer. Then, can write for any $j \in \mathbb{N}$ the integration by parts formula for functions $u, v \in C^\infty(\Gamma)$

$$\int_\Gamma (-1)^j \Delta_\Gamma^j u v \,\mathrm{d}S(x) = \int_\Gamma \Delta_\Gamma^{j/2} u \cdot \Delta_\Gamma^{j/2} v \,\mathrm{d}S(x),$$

where the dot product coincides with the usual product if j is multiple of two. With the seminorms $|\cdot|_{H^j(\Gamma)} := \|\Delta_\Gamma^{j/2} \cdot\|_{L^2(\Gamma)}$, $j = 0, \ldots, J$ we can define the Sobolev spaces $H^J(\Gamma)$ in three dimensions and $V_J := H^1(\Omega) \cap H^J(\Gamma)$.

Then, the weak formulation for (11) reads: Seek $u \in V_J$ such that

$$\int_\Omega \left(\nabla u \cdot \nabla \overline{v} - \kappa^2 u \overline{v}\right) \mathrm{d}x + \sum_{j=0}^{J} \alpha_j \int_\Gamma \Delta_\Gamma^{j/2} u \cdot \Delta_\Gamma^{j/2} \overline{v} \,\mathrm{d}S(x) = \langle f_J, v \rangle \quad \forall\, v \in V_J, \tag{13}$$

where f_J corresponds to the source terms. If $\mathrm{Re}(a_J) > 0$ or $|\mathrm{Im}(a_J)| > 0$ then the bilinear form a_J can be written as a sum of a V_J-elliptic bilinear form $\mathsf{a}_{J,0}$ and a bilinear form k with only lower derivatives corresponding to a compact operator in V_J. Hence, it exists a Gårding inequality. Then, the Fredholm alternative applies and uniqueness of (13) implies existence of a solution (similar to [46, Chap. 2]).

The usual way to incorporate conditions with derivatives on the normal trace are mixed formulations which introduce a new unknown $\lambda := \partial_n u$ and take the condition in weak form as an additional equation. We prefer to incorporate the condition (12) in addition in the original equation. Then, with the two bilinear forms

$$\mathsf{b}_{J,0}((u,\lambda),v) := \int_\Omega \left(\nabla u\cdot\nabla\overline{v} - \kappa^2 u\overline{v}\right)\mathrm{d}x + \sum_{j=0}^{J}\alpha_j\int_\Gamma \Delta_\Gamma^{j/2}u\cdot\Delta_\Gamma^{j/2}\overline{v}\,\mathrm{d}S(x) + \sum_{j=1}^{J}\beta_j\int_\Gamma \Delta_\Gamma^{j/2}\lambda\cdot\Delta_\Gamma^{j/2}\overline{v}\,\mathrm{d}S(x),$$

$$\mathsf{b}_{J,1}((u,\lambda),\lambda') := \int_\Gamma \Big(\lambda\overline{\lambda'} - \sum_{j=0}^{J}\alpha_j\Delta_\Gamma^{j/2}u\cdot\Delta_\Gamma^{j/2}\overline{\lambda'} - \sum_{j=1}^{J}\beta_j\Delta_\Gamma^{j/2}\lambda\cdot\Delta_\Gamma^{j/2}\overline{\lambda'}\Big)\mathrm{d}S(x)$$

the mixed variational formulation for the Helmholtz equation with (12) on Γ reads: Seek $(u,\lambda)\in V_J\times H^J(\Gamma)$ such that

$$\mathsf{b}_J((u,\lambda),(v,\lambda')) := \mathsf{b}_{J,0}((u,\lambda),v) + \mathsf{b}_{J,1}((u,\lambda),\lambda') = \langle f_J, v\rangle \quad \forall\,(v,\lambda')\in V_J\times H^J(\Gamma). \tag{14}$$

This system uses the product bilinear form b_J. As the two equations are independent, we can equivalently solve for $\mathsf{b}_{J,0}((u,\lambda),v) + \gamma_J\overline{\mathsf{b}_{J,1}((u,\lambda),\lambda')} = \langle f_J, v\rangle$ where the conjugate complex of the second equation is taken and $\gamma_J \neq 0$ is an arbitrary complex factor. If we use then the test functions $v = u$ and $\lambda' = \lambda$ and $\gamma_J = \frac{\beta_J}{\overline{\alpha}_J}$ the mixed terms of highest order cancel out

$$\begin{aligned}\mathsf{b}_{J,0}((u,\lambda),u) + \frac{\beta_J}{\overline{\alpha}_J}\overline{\mathsf{b}_{J,1}((u,\lambda),\lambda)} &= |u|^2_{H^1(\Omega)} + \alpha_J|u|^2_{H^J(\Gamma)} - \frac{|\beta_J|^2}{\overline{\alpha}_J}|\lambda|^2_{H^J(\Gamma)}\\ &\quad + \langle\kappa u, u\rangle_{L^2(\Omega)} + \sum_{j=0}^{J-1}\alpha_j|u|^2_{H^j(\Gamma)} + \frac{\beta_J}{\overline{\alpha}_J}\Big(\|\lambda\|_{L^2(\Gamma)} - \sum_{j=1}^{J-1}\overline{\beta}_j|\lambda|^2_{H^j(\Gamma)}\Big)\\ &\quad + \sum_{j=1}^{J-1}\Big(\beta_j - \frac{\beta_J}{\overline{\alpha}_J}\overline{\alpha}_j\Big)\big\langle\Delta_\Gamma^{j/2}\lambda, \Delta_\Gamma^{j/2}u\big\rangle_{L^2(\Gamma)} - \frac{\beta_J}{\overline{\alpha}_J}\overline{\alpha}_0\langle\lambda, u\rangle.\end{aligned}$$

If $|\mathrm{Im}(a_J)| > 0$ and $\beta_J \neq 0$ then there holds a Gårding inequality, , there exists a constant $\theta\in(-\pi,\pi)$ and

$$\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\theta}\Big(\mathsf{b}_{J,0}((u,\lambda),u) + \frac{\beta_J}{\overline{\alpha}_J}\overline{\mathsf{b}_{J,1}((u,\lambda),\lambda)}\Big)\right) \geq \gamma\left(\|u\|^2_{V_J} + \|\lambda\|^2_{H^J(\Gamma)}\right) - \delta\left(\|u\|^2_{W_{J-1}} + \|\lambda\|^2_{H^{J-1}(\Gamma)}\right),$$

for some constants $\gamma > 0$ and $\delta\in\mathbb{R}$, where $W_{J-1} := L^2(\Omega)\cap H^{J-1}(\Gamma)$. Hence, the Fredholm alternative applies as well and we a unique solution of (14) exist except for a set of spurious eigenmodes.

3.2. Definition of the C^0-continuous finite element spaces

Similarly to the two-dimensional case, the non-conforming finite element method is based on a mesh $\mathcal{M}_h$ of the computational domain Ω consisting of possibly curved tetrahedra, hexahedra, prism or pyramids, which are disjoint and which fill the computational domain, , $\Omega = \bigcup_{K\in\mathcal{M}_h}\overline{K}$. We denote $\mathcal{F}(\mathcal{M}_h,\Gamma)$ the set of faces (triangles or quadrilaterals) of $\mathcal{M}_h$ on Γ, $\mathcal{E}(\mathcal{M}_h,\Gamma)$ is the set of edges of $\mathcal{M}_h$ on Γ and $\mathcal{E}(e)$ is the set composed of all the edges of the external boundary Γ. Furthermore, we define the union of all outer boundary faces and the union of all cells as

$$\Gamma_h := \bigcup_{f\in\mathcal{F}(\mathcal{M}_h,\Gamma)} f = \Gamma\backslash\bigcup_{e\in\mathcal{E}(\mathcal{M}_h,\Gamma)} e, \qquad \Omega_h := \bigcup_{K\in\mathcal{M}_h} K = \Omega\backslash\bigcup_{f\in\mathcal{F}(\mathcal{M}_h)}\overline{f}.$$

We assume that each face has direct orientation and can be represented by a smooth mapping F_f from the reference triangle or quadrilateral $\hat{F}$. As in 2D, the mesh width h is the largest outer diameter of the cells and $V_h := S^{\mathsf{p},1}(\Omega,\mathcal{M}_h)$ is the space of piecewise polynomial, continuous basis functions with polynomial orders p. In the same way, $V_h\subset V_{j,h} := H^1(\Omega)\cap H^1(\Gamma)\cap H^j(\Gamma_h)$ for any $j\in\mathbb{N}$.

Finally, we have to define additional notations specific to the non-conforming formulation. For each edge $e \in \mathcal{E}(\mathcal{M}_h, \Gamma)$, we denote arbitrarily by f_e^+ and f_e^- the two external faces sharing e, and we define respectively by v_e^+ and by v_e^- the trace of v on e taken from within f_e^+ and f_e^-. Furthermore, on an edge e of a face $f \in \mathcal{F}(\mathcal{M}_h, \Gamma)$, we denote by τ_e the vector orthogonal to ν and e and outward to f. The jump and the mean of a scalar function v on e are respectively defined by

$$[v]_e = v_e^- - v_e^+ \quad \text{and} \quad \{v\}_e = \frac{v_e^+ + v_e^-}{2},$$

while the jump and the mean of a vectorial function v on e are respectively defined by

$$[v]_e = v_e^- \cdot \tau_e^- + v_e^+ \cdot \tau_e^+ \quad \text{and} \quad \{v\}_e = \frac{v_e^- \cdot \tau_e^- - v_e^+ \cdot \tau_e^+}{2},$$

We also denote by h_f the diameter of the face f and by $h_e = \min\left(h_{f_e^+}, h_{f_e^-}\right)$.

3.3. Definition of the interior penalty Galerkin variational formulation

The construction of the interior penalty Galerkin formulation is similar to the 2D case. First, we multiply $\Delta_\Gamma^{j/2}\alpha_j\Delta_\Gamma^{j/2}u$ for a function $u \in C^\infty$ by $\overline{v}_h \in V_h$, which is in $C^\infty(\overline{f})$ in each face f on Γ, and we apply j times integration by part in each face f. Second, we used the equivalence $[ab]_e = [a]_e\{b\}_e + \{a\}_e[b]_e$, the fact that with $u_j, \alpha_j \in C^\infty$ all jumps $\left[\Delta_\Gamma^{(j-i-1)/2}\alpha_j\Delta_\Gamma^{j/2}u\right]_e$, $i < j$ are zero and that with $v_h \in C^0(\Gamma)$ all jumps $\left[\overline{v}\right]_e$ are zero. Third, we add the terms $s\int_e[\Delta_\Gamma^{i/2}u]_e\{\Delta_\Gamma^{(j-i-1)/2}\alpha_j\Delta_\Gamma^{j/2}\overline{v}_h\}_e \,\mathrm{d}\sigma(x)$ with $s = 1$ for IPDG, $s = -1$ for NIPDG and $s = 0$ for IIPDG. We do not loose consistency as with the assumption of $u \in C^\infty(\Gamma)$ the terms $\left[\Delta_\Gamma^{i/2}u\right]_e$ are in fact zero. Finally, to ensure the coercivity of the bilinear forms (with a mesh-independent constant), we add for $j > 0$ the terms

$$\frac{\beta_j}{h_e^{2(J-j)+1}}\int_e[\Delta_\Gamma^{j-1/2}u]_e[\Delta_\Gamma^{j-1/2}\overline{v}_h]_e \,\mathrm{d}\sigma(x)$$

which do not harm the consistency since $[\Delta_\Gamma^{j-1/2}u]_e = 0$, $j = 1, \ldots, J-1$.

The interior penalty Galerkin formulation reads then: Seek $u_{J,h} \in V_h$ such that

$$\mathsf{a}_{J,h}(u_{J,h}, v_h) = \langle f_J, v_h\rangle\,, \quad \forall v_h \in V_h, \tag{15}$$

where

$$\mathsf{a}_{J,h}(u_h, v_h) := \int_\Omega \left(\nabla u \cdot \nabla\overline{v} - \kappa^2 u\overline{v}\right)\mathrm{d}x + \sum_{j=0}^{J}\left(\int_{\Gamma_h}\alpha_j\Delta_\Gamma^{j/2}u_h \cdot \Delta_\Gamma^{j/2}\overline{v}_h \,\mathrm{d}S(x) + \mathsf{b}_{j,h}(u_h, v_h)\right)$$

and $\mathsf{b}_{0,h} = \mathsf{b}_{1,h} = 0$ and for $j > 1$

$$\begin{aligned}\mathsf{b}_{j,h}(u_h, v_h) := &\sum_{i=1}^{j-1}(-1)^{i+j}\sum_{e\in\mathcal{E}(\mathcal{M}_h,\Gamma)}\int_e\left(\left\{\Delta_\Gamma^{(j-i-1)/2}\alpha_j\Delta_\Gamma^{j/2}u_h\right\}_e\left[\Delta_\Gamma^{i/2}\overline{v}_h\right]_e + s\left[\Delta_\Gamma^{i/2}u_h\right]_e\left\{\Delta_\Gamma^{(j-i-1)/2}\alpha_j\Delta_\Gamma^{j/2}\overline{v}_h\right\}_e\right)\,\mathrm{d}\sigma(x)\\ &+\sum_{e\in\mathcal{E}(\mathcal{M}_h,\Gamma)}\frac{\beta_j}{h_e^{2(J-j)+1}}\int_e[\Delta_\Gamma^{j-1/2}u_h]_e[\Delta_\Gamma^{j-1/2}\overline{v}_h]_e \,\mathrm{d}\sigma(x).\end{aligned}$$

4. Analysis of the interior penalty formulation in 2D

4.1. Associated variational formulation for infinite-dimensional spaces

Following [25] we are going to define a interior-penalty Galerkin variational formulation which is identical to discrete one for the discrete space V_h and which can be defined for infinite-dimensional function spaces as well. As the discrete space V_h is not contained in the continuous function spaces V_J for

$J \geq 2$, we will use larger spaces $V_{J,h}$, which include both V_h and V_J. These infinite-dimensional spaces are defined by

$$V_{J,h} := H^1(\Omega) \cap H^1(\Gamma) \cap H^J(\Gamma_h) \supset V_J \tag{16}$$

with the norm

$$\|v\|^2_{V_{J,h}} := \|v\|^2_{H^1(\Omega)} + \|v\|^2_{H^J(\Gamma_h)} + \sum_{j=2}^{J} \sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)} \frac{1}{h_n^{2(J-j)+1}} |\left[\partial_\Gamma^{j-1} v\right]_n|^2.$$

Note, that the trace of functions $v \in V_{J,h}$ is continuous on Γ and their tangential derivatives of order 1 to $J-1$ are bounded, but may be discontinuous over the nodes $\mathcal{N}(\mathcal{M}_h, \Gamma)$.

The discrete formulation (6) cannot directly be used with the function space $V_{J,h}$ as the terms $\left\{\partial_\Gamma^{j-i-1}\alpha_j\partial_\Gamma^j v\right\}_n$ for $v \in V_{J,h}$ are not well-defined for $2j - i > J$. In the discrete variational formulation those terms occur only in product with the finitely many piecewise polynomial functions in V_h. We can define an extension of these products using the lifting operators $\mathcal{L}_{j,i} : H^1(\Gamma_h) \to V_h$, $i, j \in \mathbb{N}$ by

$$\int_{\Gamma_h} \mathcal{L}_{j,i}(v)\alpha_j\partial_\Gamma^j w_h \,\mathrm{d}\sigma(x) = \sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)} [v]_n\{\partial_\Gamma^{j-i-1}\alpha_j\partial_\Gamma^j w_h\}_n \quad \forall w_h \in V_h, \tag{17}$$

and replace any occurance of $\sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)} \left\{\partial_\Gamma^{j-i-1}\alpha_j\partial_\Gamma^j u_h\right\}_n \left[\partial_\Gamma^i \overline{v}_h\right]_n$ or its symmetric counterpart in the discrete formulation (6) by $\int_{\Gamma_h} \alpha_j\partial_\Gamma^j u\mathcal{L}_{j,i}(\partial_\Gamma^i\overline{v}) \,\mathrm{d}\sigma(x)$ or its symmetric counterpart, respectively, in that for the infinite-dimensional spaces.

Now, we can define the interior penalty Galerkin variational formulation for the infinite-dimensional spaces $V_{J,h}$: Seek $\widetilde{u}_J \in V_{J,h}$ such that

$$\widetilde{\mathsf{a}}_{J,h}(\widetilde{u}_J, v) = \langle f_{J,h}, v\rangle, \quad \forall v \in V_{J,h}, \tag{18}$$

where

$$\widetilde{\mathsf{a}}_{J,h}(u, v) := \int_\Omega \left(\nabla u_J \cdot \nabla\overline{v} - \kappa^2 u_J\overline{v}\right) \,\mathrm{d}x + \sum_{j=0}^{J} \Big(\mathsf{c}_j(u_h, v_h; \alpha_j) + \widetilde{\mathsf{b}}_{j,h;J}(u, v; \alpha_j)\Big)$$

and for $\widetilde{\mathsf{b}}_{0,h;J} = \widetilde{\mathsf{b}}_{1,h;J} = 0$ and for $j > 1$

$$\begin{aligned}\widetilde{\mathsf{b}}_{j,h;J}(u, v; \alpha_j) := &\sum_{i=1}^{j-1}(-1)^{i+j}\int_\Gamma \alpha_j\mathcal{L}_{j,i}(\partial_\Gamma^i u)\partial_\Gamma^j\overline{v} + s\alpha_j\mathcal{L}_{j,i}(\partial_\Gamma^i\overline{v})\partial_\Gamma^j u \,\mathrm{d}\sigma(x)\\ &+ \sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)} \frac{\beta_j}{h_n^{2(J-j)+1}}\left[\partial_\Gamma^{j-1}u\right]_n\left[\partial_\Gamma^{j-1}\overline{v}\right]_n.\end{aligned}$$

Note, that due to the definition of the lifting operators $\widetilde{\mathsf{b}}_{j,h;J} = \mathsf{b}_{j,h;J}$ on $V_h \times V_h$ and $\widetilde{\mathsf{b}}_{j,h;J} = 0$ on $V_j \times V_j$ as all jump terms and so all lifting operators vanish and the weak derivatives exists on whole Γ, not only on Γ_h. Hence, $\widetilde{\mathsf{a}}_{J,h} = \mathsf{a}_{J,h}$ on $V_h \times V_h$ and $\widetilde{\mathsf{a}}_{J,h} = \mathsf{a}_J$ on $V_J \times V_J$.

4.2. Analysis of the associated variational formulation

We will need in the following the equivalence of (3) and (18).

Lemma 4.1. *Let $\langle f_J, v\rangle = \int_\Omega fv \,\mathrm{d}x + \int_\Gamma gv \,\mathrm{d}\sigma(x)$ with $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$. Then, the formulations (3) and (18) possess the same solutions, , if $u_J \in V_J$ is solution of (3), then it solves (18), and if $\widetilde{u}_J \in V_{J,h}$ is solution of (18), then it solves (3).*

Proof. If $J = 0, 1$ the formulations (3) and (18) are identical, whereby we restrict ourself to $J \geq 2$. The proof is in two steps. First, we prove that the solution $u_J \in V_J$ of (3) solves (18), and then, that the solution $\widetilde{u}_J \in V_{J,h}$ of (18) solves (3).

(i) Let $u_J \in V_J$ solution of (3). Taking test functions $v \in H_0^1(\Omega) \supset C_c^\infty(\Omega)$ vanishing on $\partial\Omega$ in (3) we can assert using the definition of weak derivatives that u_J solves

$$-\Delta u_J - \kappa^2 u_J = f, \quad \text{in } \Omega. \tag{19a}$$

In case of non-empty $\partial\Omega\backslash\Gamma$ we take test functions $v \in H^1(\Omega)$ with $v \equiv 0$ on Γ, and using integration by parts in Ω and the fact that u_J solves (19a) we find that u_J solves

$$\partial_\nu u_J = 0 \quad \text{on } \partial\Omega\backslash\Gamma. \tag{19b}$$

Now, taking test functions v in the whole space V_J, using integration by parts in Ω and on Γ, and using the fact that u_J solves (19a) and (19b) we find in the same way

$$\partial_\nu u_J + \sum_{j=0}^{J} (-1)^j \partial_\Gamma^j (\alpha_j \partial_\Gamma^j u_J) = g, \quad \text{on } \Gamma. \tag{19c}$$

Following the same steps as for the construction of the bilinear form $a_{J,h}$ (but using the lifting operators instead of the jump terms), it follows easily that u_J solves $\widetilde{\mathsf{a}}_{J,h}(u_J, v) = \langle f_{J,h}, v\rangle$ for all $v \in V_{J,h}$.

(ii) Let $\widetilde{u}_J \in V_{J,h}$ solution of (18). In the same way as in Part (i) we find that $\widetilde{u}_J$ solves (19a) and (19b). Now, we take test functions $v \in V_J \cap C_c^\infty(\Gamma_h)$ such that $\mathsf{b}_j(u,v) = \widetilde{\mathsf{b}}_{j,h}(u,v)$ holds for any $u \in V_J$. Then, using integration by parts in Ω and the fact that $\widetilde{u}_J$ solves (19a) and (19b) we find that $\widetilde{u}_J$ solves (19c) on Γ_h.

Comparing with the integration by parts formula in (4) we find that for all $v \in V_J \cap C^\infty(\Gamma_h)$

$$\sum_{j=2}^{J} \sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)} \left(\sum_{i=1}^{j-1} (-1)^{i+j} \left(\left[\partial_\Gamma^{j-i-1}\alpha_j \partial_\Gamma^j \widetilde{u}_J\right]_n \left\{\partial_\Gamma^i \overline{v}\right\}_n - s\left[\partial_\Gamma^i \widetilde{u}_J\right]_n \left\{\partial_\Gamma^{j-i-1}\alpha_j \partial_\Gamma^j \overline{v}\right\}_n \right) \right.$$
$$\left. + \frac{\beta_j}{h_n^{2(J-j)+1}} \left[\partial_\Gamma^{j-1}\widetilde{u}_J\right]_n \left[\partial_\Gamma^{j-1}\overline{v}\right]_n \right) = 0.$$

If we take test functions $v \in V_J \cap C^\infty(\Gamma_h)$ for which it holds $\left\{\partial_\Gamma^{j-1}\overline{v}\right\}_n = 0$ for all $j = 2, \ldots, J$ and all $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$ and for which $\left\{\partial_\Gamma^{j-i-1}\alpha_j \partial_\Gamma^j \overline{v}\right\}_n = 0$ for all $j = 2, \ldots, J$ and $i = 1, \ldots, j-1$ if $s \neq 0$ then it has to hold that

$$\sum_{j=2}^{J} \sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)} \frac{\beta_j}{h_n^{2(J-j)+1}} \left[\partial_\Gamma^{j-1}\widetilde{u}_J\right]_n \left[\partial_\Gamma^{j-1}\overline{v}\right]_n = 0.$$

This is only possible if $\left[\partial_\Gamma^{j-1}\widetilde{u}_J\right]_n = 0$ for $j = 2, \ldots, J$ and any $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$.

Hence, we have shown that $\widetilde{u}_J$ is in V_J and solves (19) and so (3).

This completes the proof. □

In order to prove the well-posedness of the variational formulation (18) we need the following lemmata.

Lemma 4.2. *The lifting operators $\mathcal{L}_{j,i} : H^1(\Gamma_h) \to V_h$, $i,j \in \mathbb{N}$ defined by (17) are continuous, , there exists constants $C_{j,i} > 0$ such that*

$$\|\mathcal{L}_{j,i}(v)\|^2_{L^2(\Gamma)} \leq C_{j,i}^2 \sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)} \frac{|[v]_n|^2}{h_n^{2(j-i)-1}}.$$

Proof. From the definition of the mean over n, we deduce

$$\left|\left\{\partial_\Gamma^{j-i-1}\alpha_j\partial_\Gamma^j w_h\right\}_n\right|^2 \leq \frac{\left|(\partial_\Gamma^{j-i-1}\alpha_j\partial_\Gamma^j w_h)_n^+\right|^2 + \left|(\partial_\Gamma^{j-i-1}\alpha_j\partial_\Gamma^j w_h)_n^-\right|^2}{2}, \tag{20}$$

with the notation $(v)_n^\pm = v_n^\pm$. Now, using inverse inequalities [50, 58] and the fact that $h_n \leq h_{e_n^\pm}$ we obtain

$$\left|(\partial_\Gamma^{j-i-1}\alpha_j\partial_\Gamma^j w_h)_n^\pm\right|^2 \leq \frac{C_{j,i}^2}{2\,h_n^{2(j-i)-1}}\int_{e_n^\pm}|\alpha_j\partial_\Gamma^j w_h|^2\,\mathrm{d}\sigma(x) \tag{21}$$

with $C_{j,i} := C_{j,i}(p_\Gamma) = O(p_\Gamma^{2(j-i)-1})$, and using (20) we can assert that

$$\left|\left\{\partial_\Gamma^{j-i-1}\alpha_j\partial_\Gamma^j w_h\right\}_n\right|^2 \leq \frac{C_{j,i}^2}{2\,h_n^{2(j-i)-1}}\int_{e_n^+\cup e_n^-}|\alpha_j\partial_\Gamma^j w_h|^2\,\mathrm{d}\sigma(x)$$

and so

$$\sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)} h_n^{2(j-i)-1}\left|\left\{\partial_\Gamma^{j-i-1}\alpha_j\partial_\Gamma^j w_h\right\}_n\right|^2 \leq C_{j,i}^2\|\alpha_j\partial_\Gamma^j w_h\|_{L^2(\Gamma)}^2. \tag{22}$$

Now, using the definition of the lifting operator $\mathcal{L}_{j,i}$ and the Cauchy-Schwarz-inequality, we have

$$\begin{aligned}\|\mathcal{L}_{j,i}(v)\|_{L^2(\Gamma)}^2 &= \max_{w_h\in V_h}\frac{\left|\displaystyle\sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)}[v]_n\left\{\partial_\Gamma^{j-i-1}\alpha_j\partial_\Gamma^j w_h\right\}_n\right|^2}{\|\alpha_j\partial_\Gamma^j w_h\|_{L^2(\Gamma)}^2}\\ &\leq \max_{w_h\in V_h}\frac{\displaystyle\sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)}\frac{|[v]_n|^2}{h_n^{2(j-i)-1}}\sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)}h_n^{2(j-i)-1}\left|\left\{\partial_\Gamma^{j-i-1}\alpha_j\partial_\Gamma^j w_h\right\}_n\right|^2}{\|\alpha_j\partial_\Gamma^j w_h\|_{L^2(\Gamma)}^2},\end{aligned}$$

and inserting (22) we conclude in the statement of the lemma. □

Lemma 4.3. *Let Let $s\in[-1,1]$, $J\geq 1$, $\alpha_J\in L^\infty(\Gamma)$ with $\inf_{x\in\Gamma}\mathrm{Re}(\alpha_J)>0$ or $\inf_{x\in\Gamma}|\mathrm{Im}(\alpha_J)|>0$. Then, for β_j large enough the bilinear form $\widetilde{\mathsf{a}}_{0,J}$ defined by*

$$\widetilde{\mathsf{a}}_{0,J}(u,v) := \int_\Omega(\nabla u\cdot\nabla\overline{v}+u\overline{v})\,\mathrm{d}x + \sum_{j=0}^{J-1}\Big(\mathsf{c}_{j,h}(u,v;1)+\widetilde{\mathsf{b}}_{j,h;J}(u,v;1)\Big) + \mathsf{c}_{J,h}(u,v;1)+\widetilde{\mathsf{b}}_{J,h;J}(u,v;\alpha_J), \tag{23}$$

is $V_{J,h}$-elliptic with an ellipticity constant independent of h_n for all $n\in\mathcal{N}(\mathcal{M}_h,\Gamma)$.

The proof is a simple consequence of the following lemma, in which we give a more explicit statement how large the penalty terms β_j need to be.

Lemma 4.4. *Let the assumption on s and α_J in Lemma 4.3 be fulfilled. Then, there exist constants C, γ and θ, such that for $\beta_j>\sqrt{j}\,C$, $j\geq 2$ it holds*

$$\begin{aligned}\mathrm{Re}\Big(\mathrm{e}^{\mathrm{i}\theta}\sum_{j=0}^{J-1}\Big(\mathsf{c}_{j,h}(v,v;1)+\widetilde{\mathsf{b}}_{j,h;J}(v,v;1)\Big)\Big)+\mathrm{Re}\Big(\mathrm{e}^{\mathrm{i}\theta}\Big(\mathsf{c}_{J,h}(v,v;\alpha_J)+\widetilde{\mathsf{b}}_{J,h;J}(v,v;\alpha_J)\Big)\Big)&\\ \geq\gamma\left(\|v\|_{H^J(\Gamma)}^2+\sum_{j=2}^{J}\sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)}\frac{1}{h_n^{2(J-j)+1}}\left|\left[\partial_\Gamma^{j-1}v\right]_n\right|^2\right)\quad&\forall v\in H^j(\Gamma_h)\,.\end{aligned}$$

Proof. With the assumption on α_J there exists $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that with a postive constant γ_J it holds

$$\operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\theta}\int_\Gamma \alpha_J |v|^2\right) \geq \gamma_J |v|^2_{L^2(\Gamma)}. \tag{24}$$

In the remainder of the proof we assume θ such that (24) is fulfilled.

We start the proof by writing

$$\operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\theta}\widetilde{\mathsf{b}}_{j,h;J}(v,v;\alpha_j)\right) = \sum_{i=1}^{j-1}(-1)^{i+j}\operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\theta}\int_\Gamma \alpha_j\mathcal{L}_{j,i}(\partial^i_\Gamma v)\partial^j_\Gamma\overline{v} + s\alpha_j\mathcal{L}_{j,i}(\partial^i_\Gamma\overline{v})\partial^j_\Gamma v\,\mathrm{d}\sigma(x)\right)$$
$$+ \cos(\theta)\sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)}\frac{\beta_j}{h_n^{2(J-j)+1}}\left|\left[\partial_\Gamma^{j-1}v\right]_n\right|^2.$$

Using that $2ab \leq \varepsilon a^2 + \varepsilon^{-1}b^2$ for any positive ε, the fact that $\alpha_j = 1$ for $j < J$ in the bilinear forms $\widetilde{\mathsf{b}}_{j,h;J}$, the assumption on α_J and Lemma 4.2, we obtain

$$2\left|\int_\Gamma \alpha_j\mathcal{L}_{j,i}(\partial^i_\Gamma v)\partial^j_\Gamma\overline{v}\,\mathrm{d}\sigma(x)\right| \leq \varepsilon\|\partial^j_\Gamma v\|^2_{L^2(\Gamma)} + \varepsilon^{-1}\|\alpha_j\mathcal{L}_{j,i}(\partial^i_\Gamma v)\|^2_{L^2(\Gamma)}$$
$$\leq \varepsilon\|\partial^j_\Gamma v\|^2_{L^2(\Gamma)} + \varepsilon^{-1}\|\alpha_j\|^2_{L^\infty(\Gamma)}C^2_{j,i}\sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)}\frac{1}{h_n^{2(j-i)-1}}|\left[\partial^i_\Gamma v\right]_n|^2,$$

where the same bound holds, if v and $\overline{v}$ are interchanged. Hence, we can assert that

$$\operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\theta}\sum_{j=0}^{J-1}\left(\mathsf{c}_{j,h}(v,v;1) + \widetilde{\mathsf{b}}_{j,h;J}(v,v;1)\right)\right) + \operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\theta}\left(\mathsf{c}_{J,h}(v,v;\alpha_J) + \widetilde{\mathsf{b}}_{J,h;J}(v,v;\alpha_J)\right)\right)$$
$$\geq \cos(\theta)\|v\|^2_{H^1(\Gamma)} + \sum_{j=2}^{J-1}\left(\cos(\theta) - (j-1)\,\varepsilon_j\right)\|\partial^j_\Gamma v\|^2_{L^2(\Gamma)} + \left(\gamma_J - (J-1)\,\varepsilon_J\right)\|\partial^J_\Gamma v\|^2_{L^2(\Gamma)}$$
$$+\sum_{j=2}^{J}\sum_{n\in\mathcal{N}(\mathcal{M}_h,\Gamma)}\left(\frac{\cos(\theta)\beta_j}{h_n^{2(J-j)+1}} - \sum_{i=j}^{J}\varepsilon_i^{-1}\|\alpha_i\|_{L^\infty(\Gamma)}\frac{C^2_{i,j-1}}{h_n^{2(i-j)+1}}\right)\left|\left[\partial^{j-1}_\Gamma v\right]_n\right|^2.$$

Finally, choosing $\varepsilon_j = \cos(\theta)/j$ for $j = 2, \ldots, J-1$ and $\varepsilon_J = \gamma_J/J$, and with the assumption on β_j we obtain the desired inequality. ☐

We are going to state here the main results for the variational formulation (18), where we will prove the well-posedness of (18) later.

Theorem 4.5. *Let the assumption of Lemma 4.4 be satisfied, and let zero be the only solution of (18) with $u^{\mathrm{inc}} \equiv 0$. Then, there exists a unique solution $\widetilde{u}_J \in V_{J,h}$ of (18) and there exists a constant $C_J > 0$ independent of h_n for all $n \in \mathcal{N}(\mathcal{M}_h,\Gamma)$ such that*

$$\|\widetilde{u}_J\|_{V_{J,h}} \leq C_J\|f_{J,h}\|_{V'_J}.$$

Proof. The proof is along the lines of that of [46, Theorem 2.3]. By Lemma 4.3 the bilinear form $\widetilde{\mathsf{a}}_{0,J}$ is $V_{J,h}$-elliptic and so the associated operators $\widetilde{A}_{0,J}$ are isomorphism in $V_{J,h}$. We define the Sobolev spaces

$$W_{0,h} := L^2(\Omega), \qquad\qquad W_{J,h} := L^2(\Omega)\cap H^{J-1}(\Gamma_h), \quad J > 0,$$

and the Rellich-Kondrachov compactness theorem [1, Chap. 6] implies that the embedding $V_{J,h} \subset\subset W_{J,h}$ is compact. Now, we define the bilinear forms

$$\widetilde{\mathsf{k}}_J(u,v) := -\int_\Gamma(\kappa^2+1)u\overline{v}\,\mathrm{d}x + \sum_{j=0}^{J-1}\left(\mathsf{c}_{j,h}(u,v;\alpha_j - 1) + \widetilde{\mathsf{b}}_{j,h;J}(u,v;\alpha_j-1)\right), \quad J > 0,$$

and their associated operators $\widetilde{K}_J$ are compact. Hence, the operators $\widetilde{A}_{0,J}+\widetilde{K}_J$ associated to the bilinear forms $\widetilde{\mathsf{a}}_J = \widetilde{\mathsf{a}}_{0,J} + \widetilde{\mathsf{k}}_J$ are Fredholm with index 0 and by the Fredholm alternative [43, Sec. 2.1.4] the uniqueness of a solution of (3) implies its existence and continuous dependence on the right hand side with constants independent of h_n for all $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$, and the proof is complete. □

Theorem 4.6. *Let the assumption of Lemma 4.5 be satisfied, and let $c^{-1} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ fixed with $c(x) = c_0 > 0$ for $|x| > R_C$ and $\kappa(x) = \omega/c(x)$ with the frequency $\omega > \mathbb{R}^+$. Then, (18) has a unique solution except for a countable (possibly finite) set of frequencies ω, the* spurious eigenfrequencies*, which accumulates only at infinity. The set of these frequencies coincides with those of (3).*

Proof. The statement follows in analogy of the proof of [46, Lemma 2.6] and using Lemma 4.1. □

Remark 4.7. *As the solutions $\widetilde{u}_J \in V_{J,h}$ of (18) coincides with $u_J \in V_J$ of (3) and as the eigenfrequencies coincide, the guaranteed uniqueness in case of Feng's absorbing boundary conditions for large enough domains by [46, Lemma 2.7] apply to $\widetilde{u}_J$ as well.*

4.3. Analysis of the discrete discontinuous Galerkin variational formulation

The discrete discontinuous Galerkin variational formulation (6) is the Galerkin discretisation of the associated variational formulation (18), when using V_h as the finite-dimensional subspace of $V_{J,h}$.

Proof of Theorem 2.3. By the assumption that zero is the only solution of the continuous variational formulation (3) with zero sources, we can choose a function $c(x)$ and a frequency ω such that $\kappa(x) = \omega/c(x)$ as in [46, Lemma 2.6] and ω is not a spurious eigenfrequency. For $J = 0$ and $|\mathrm{Im}\,\alpha_0| > 0$ the statement of the theorem has been proved by Melenk and Sauter [35, Thm. 5.8].

The discrete system (6) is the Galerkin discretisation of (18), which has with $\kappa(x) = \omega/c(x)$ by Theorem 4.6 the same, eigenfrequencies as (3). Both systems are non-linear in ω and we regard them in a similar fix-point form as in [46, Eq. (2.6)]. These systems are linear eigenvalue problems in ω^2 for given parameter $\widetilde{\omega} \in \mathbb{C}\backslash\{0\}$, where the fix-point system of (18) admits a countable set of frequencies $\omega_m(\widetilde{\omega})$ and that of (6) a finite set $\omega_{m,h}(\widetilde{\omega})$. As ω is not a spurious eigenfrequency, $\omega_m(\omega) \neq \omega$ for any $m \in \mathbb{N}$, and so the distances of the curves $\omega_m(\widetilde{\omega})$ to the point $\widetilde{\omega} = \omega$ is positive. Let d_m, $m \in \mathbb{N}$ denote these distances.

By the Babuška-Osborn theory [5] the discrete eigenfrequencies, $\omega_{m,h}(\widetilde{\omega})$ tend to $\omega_m(\widetilde{\omega})$ if the mesh-widths tend to zero for a minimal polynomial degree, which depends on J, or if the polynomial degrees tend to infinity. As a consequence, the distance $d_{m,h}$ of the curve $\omega_{m,h}(\widetilde{\omega})$ to the point $\widetilde{\omega} = \omega$ tends to d_m, and for a fine enough mesh or large enough polynomial degrees $|d_{m,h} - d_m| < \frac{1}{2} d_m$, and so $d_{m,h} > 0$. This means that ω is not an eigenfrequency of the discrete variational problem (6). Hence, it admits a unique solution [43, Sec. 2.1.6] bounded by (7) and, as the bilinear form satisfies a Gårding inequality, the Galerkin method is asymptotically quasi-optimal, see [11, 44] and [35, Sec. 3.2].

This completes the proof. □

Proof of Lemma 2.4. We start to estimate the error of $Q_{\Gamma,h} u_J$, where $Q_{\Gamma,h}$ is the $H^J(\Gamma_h)$-projection on Γ onto the trace space $TV_h(\Gamma)$ of V_h on Γ, which is defined by

$$\sum_{j=0}^{J} \sum_{e \in \mathcal{E}(\mathcal{M}_h,\Gamma)} \int_e \partial_\Gamma^j (Q_{\Gamma,h} u_J - u_J) \partial_\Gamma^j v_h \,\mathrm{d}\sigma(x) = 0 \quad \forall v_h \in TV_h(\Gamma).$$

By Cea's lemma [10] we can assert that

$$\|Q_{\Gamma,h} u_J - u_J\|_{H^J(\Gamma_h)} \le \inf_{v_h \in TV_h(\Gamma)} \|v_h - u_J\|_{H^J(\Gamma_h)} \le C h^{p_\Gamma - J + 1} \|u_J\|_{H^{p_\Gamma+1}(\Gamma)} \le C\, h^{p_\Gamma - J+1} \|f_J\|_{V_J'}. \tag{25}$$

Here, we used [46, Lemma 2.8] to obtain the last inequality.

Now, we define a projection $Q_{\Omega,h} : V_{J,h} \to V_h$ such that $Q_{\Omega,h} \cdot |_\Gamma = Q_{\Gamma,h}$. For u_J it is given by

$$\int_\Omega \nabla(Q_{\Omega,h} u_J - u_J) \cdot \nabla v_h + (Q_{\Omega,h} u_J - u_J)\, v_h \,\mathrm{d}x = 0 \quad \forall v_h \in V_{h,0},$$

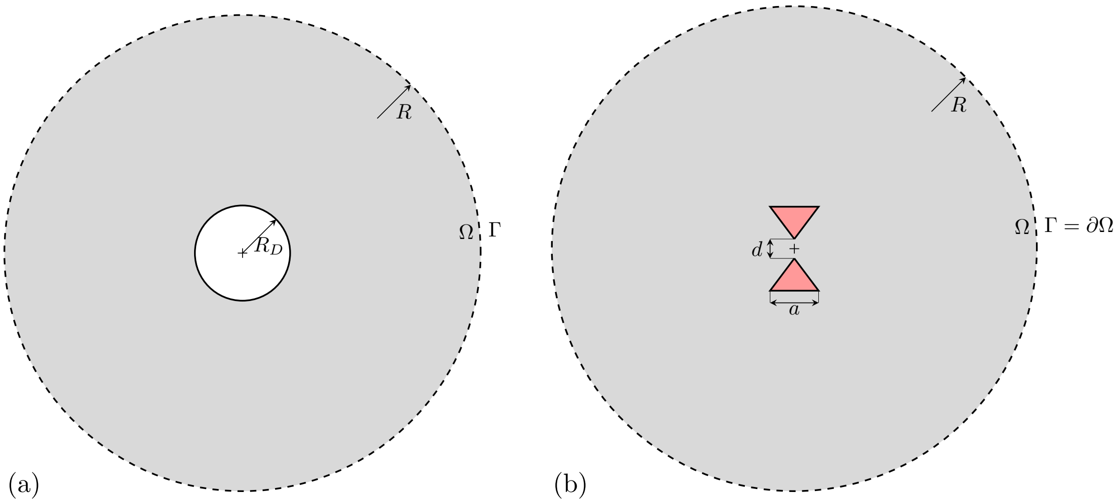

Figure 2. The geometrical setting for the (a) acoustic scattering on a rigid cylinder of radius $R_D = 1$, and (b) for the electromagnetic scatting on two cylinders with equilateral triangles as cross-section of length $a = 1.05$ and distance $d = 0.25$. The outer boundary is a circle of radius R.

where $V_{h,0}$ is the subset of functions in V_h with vanishing trace on Γ. As the projection $Q_{\Omega,h}$ is defined via the $H^1(\Omega)$-inner product its continuity follows by Lax-Milgram's lemma

$$\|Q_{\Omega,h}u_J\|_{H^1(\Omega)} \leq \|u_J\|_{H^1(\Omega)}.$$

As $Q_{\Omega,h}$ is a projection onto V_h we find that

$$\begin{aligned}\|Q_{\Omega,h}u_J - u_J\|_{H^1(\Omega)} &\leq \inf_{v_h \in V_h} \|Q_{\Omega,h}(u_J - v_h) - (u_J - v_h)\|_{H^1(\Omega)} \\ &= \inf_{v_h \in V_h} \|(Q_{\Omega,h} - Id)(u_J - v_h)\|_{H^1(\Omega)} \leq 2 \inf_{v_h \in V_h} \|u_J - v_h\|_{H^1(\Omega)}.\end{aligned}$$

Using these results we can eventually estimate

$$\begin{aligned}\Big(\inf_{v_h \in V_h} \|v_h - u_J\|_{V_{J,h}}\Big)^2 &= \inf_{v_h \in V_h} \Big(\|v_h - u_J\|^2_{H^1(\Omega)} + \sum_{j=1}^{J} |v_h - u_J|^2_{H^j(\Gamma)}\Big) \\ &\leq \|Q_{\Omega,h}u_J - u_J\|^2_{H^1(\Omega)} + \sum_{j=1}^{J} |Q_{\Gamma,h}u_J - u_J|^2_{H^j(\Gamma)} \\ &\leq 2 \inf_{w_h \in V_h} \|w_h - u_J\|^2_{H^1(\Omega)} + \|Q_{\Gamma,h}u_J - u_J\|^2_{H^J(\Gamma)}\end{aligned}$$

and using (8) the statement of the lemma follows. □

5. Numerical experiments

We have implemented the non-conforming Galerkin formulation introduced in Sec. 2 for the Feng-4 and Feng-5 conditions in the numerical C++ library Concepts [13, 19, 47] , as well as Feng-0 till Feng-3 with the usual continuous formulations (compare [57] for implementational details related to BGT absorbing boundary conditions). The *hp*-FEM part of Concepts is based on quadrilateral, curved cells in 2D where the polynomial degree can be set independently in each cell and even anisotropically. With cells having circular edges the circular boundary can be exactly resolved (see Fig. 3), where a geometry error appears only in the numerical quadrature of the integrals.

For the numerical experiments we study two model problems:

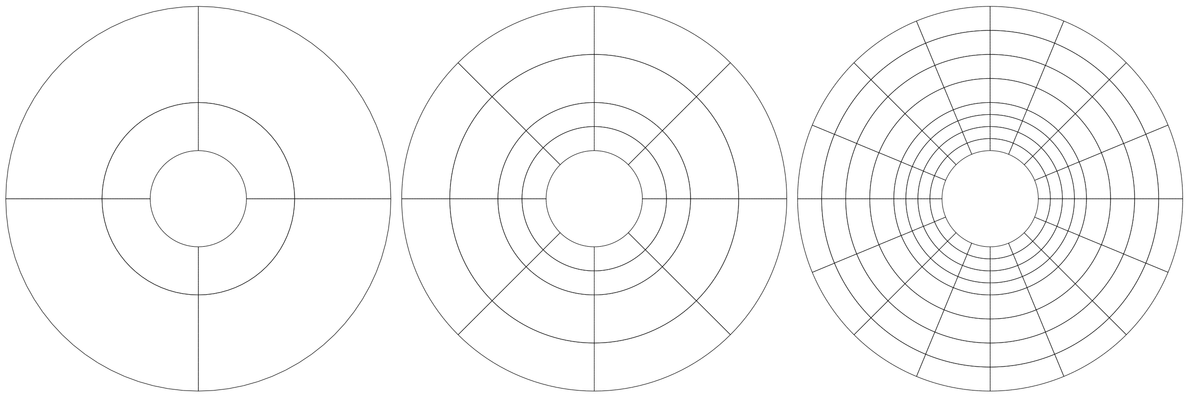

Figure 3. A sequence of curved quadrilateral meshes for the scattering on a circular disk in Concepts.

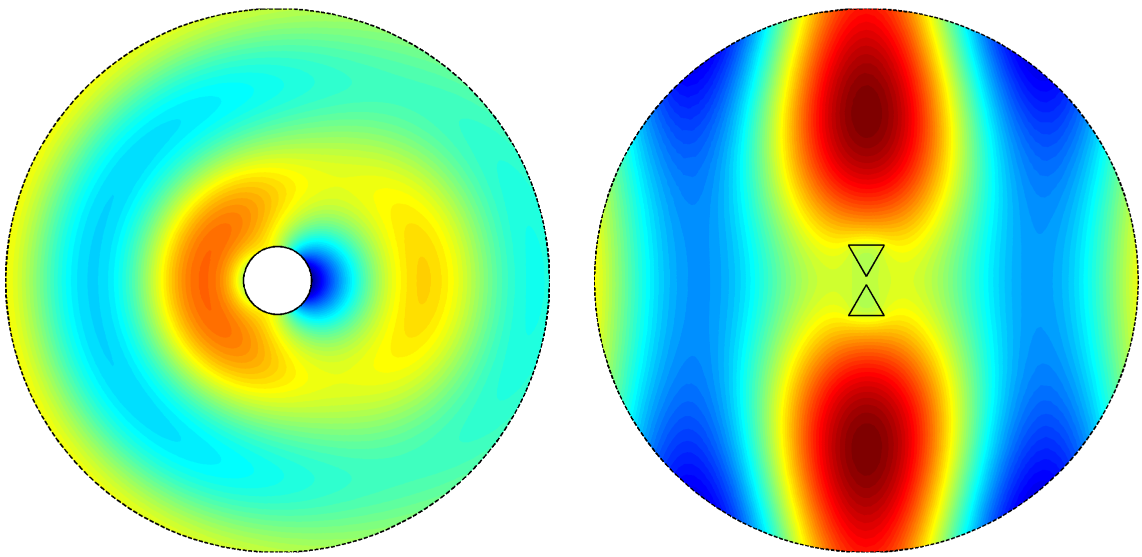

Figure 4. The scattered field (real part) for the model problem A. (left) and model problem B. (right), both with $R = 8$.

A. the acoustic scattering on a rigid cylinder with circular cross-section, where the computational domain Ω is the disk of radius R without the disk of radius $R_D = 1$ (see Fig. 2(a)) and $k = 1$, and
B. the electromagnetic scattering on two dielectric cylinders, whose cross-section are equilateral triangles of length $a = 1.05$ and distance $d = 0.25$ (see [31] and Fig. 2(b)). We have $\kappa^2(x) = \varepsilon(x)\omega^2$ with the angular frequency ω and the (relative) dielectricity $\varepsilon(x)$, which is $-40.2741 - 2.794\mathrm{i}$ inside the cylinders and 1 outside, hence $k = \omega$. We choose as frequency $\omega = 0.638$ corresponding to a wave-length in the exterior $\lambda = 2\pi/k = 9.84$.

For both model problems the incident wave is a plane wave in direction $(1, 0)^\top$ (from left). For model problem B the mesh is refined close to the nodes of the triangles.

Discretisation error for model problem A. We study the the discretisation error for the Feng-5 condition for the model problem A, where we compute on a family of meshes of the computational domain Ω with $R = 8$, see Fig. 3. Reference solutions are computed for the same model with the Feng-5 condition and on the same mesh, respectively, but with a polynomial degree which is that high, that the discretisation error of the reference solution can be neglected. The discretisation error is computed as difference of the discrete solution and the reference solution.

The results of the convergence analysis are shown in Fig. 6. We observe convergence orders of the discretisation error in the $H^1(\Omega)$-seminorm of 1.0 for $p = 1$, 2.0 for $p = 2$ and 3.0 for $p = 3$, and in the $L^2(\Omega)$-norm of 2.0 for $p = 1$, 3.0 for $p = 2$ and 4.0 for $p = 3$. Hence, we observe convergence orders that

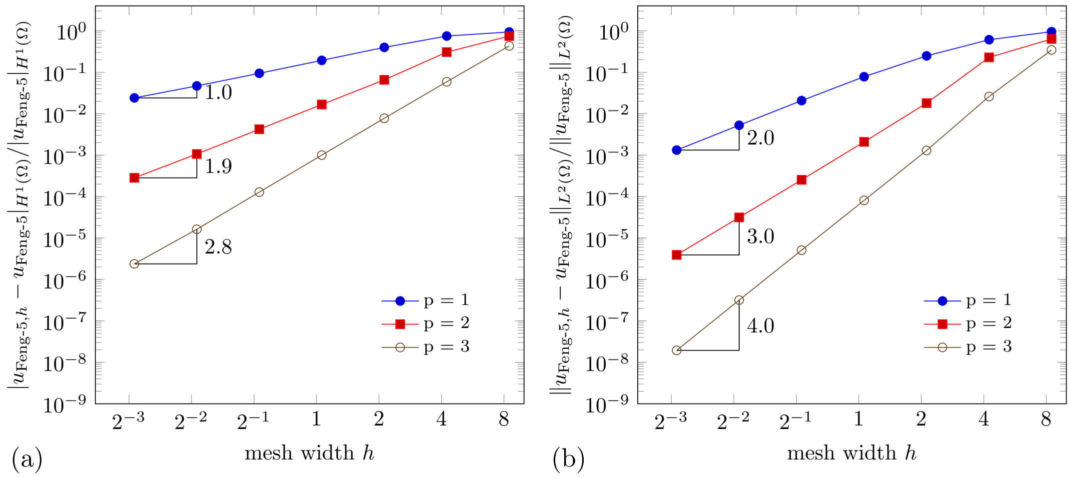

Figure 5. Convergence of the relative discretisation error in (a) the $H^1(\Omega)$-seminorm, and (b) the $L^2(\Omega)$-norm for the nonconforming Galerkin formulation with polynomial order $p = 1$ to $p = 3$ for the Feng-5 conditions for the model problem A, where $R_D = 1$ and $R = 8$.

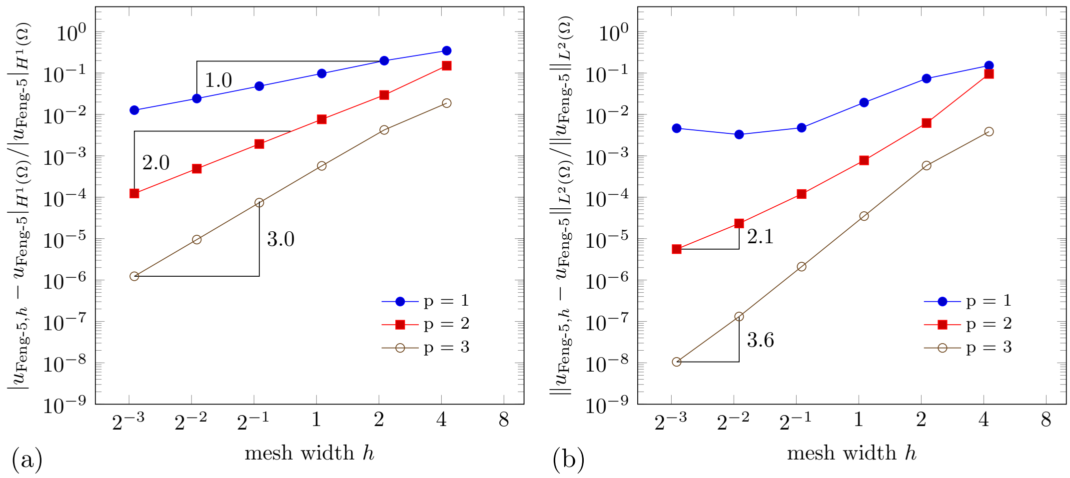

Figure 6. Convergence of the relative discretisation error in (a) the $H^1(\Omega)$-seminorm, and (b) the $L^2(\Omega)$-norm for the nonconforming Galerkin formulation with polynomial order $p = 1$ to $p = 3$ for the Feng-5 conditions for the model problem A, where $R_D = 1$ and $R = 3$.

meet the orders of the best-approximation error. In the variational formulation with Feng's conditions we have integrals of the trace of the solution and its derivatives on the outer boundary Γ. Therefore, we studied the convergence of the discretisation error on Γ as well. In the $H^1(\Gamma)$-seminorm the obtained convergence rates are 1.2, 2.0 and 3.0 for $p = 1, 2, 3$, respectively, and correlate to the convergence orders of the best-approximation error. In the $L^2(\Gamma)$-norm we get convergence rates of 2.0, 4.0, and 5.3 for $p = 1, 2, 3$, respectively. The observed convergence rates are for $p = 2$ and $p = 3$ better than the those for the best-approximation error of an arbitrary, smooth enough function.

Modelling error for model problem B. For model problem B we compare the use of Feng's conditions of different order for a fine mesh with polynomial $p = 6$, for which the discretisation error is less than $1 \cdot 10^{-6}$ in $L^\infty(\Omega)$. Hence, the modelling error is dominating. In Fig. 8 we show the modelling error for $R = 8$ using the Feng-0 condition, which is of Robin type, the Feng-2 condition, which is of Wentzel type, and the Feng-4 condition. Increasing the order of the condition leads to a significant error reduction, the error diminishes by a factor of 100 when using Feng-2 instead of Feng-0, and by another factor of

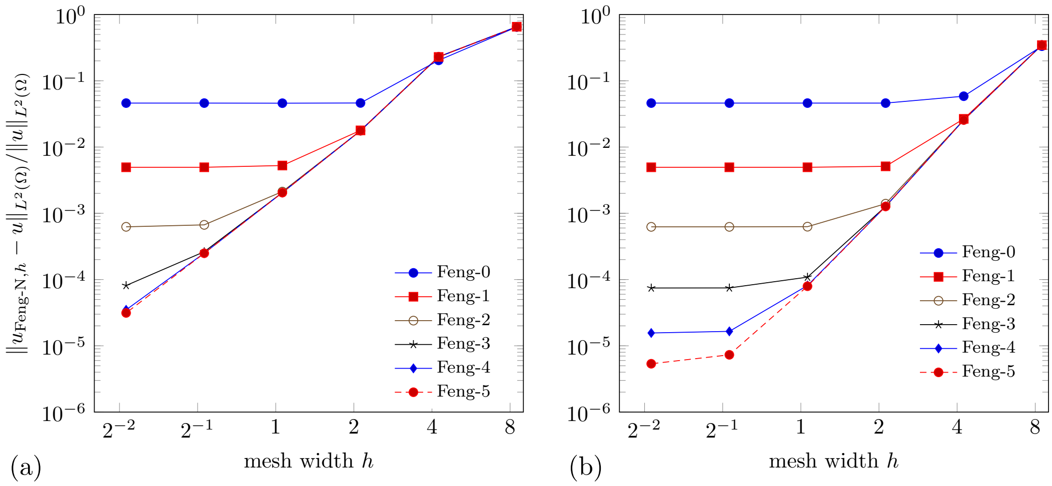

Figure 7. Convergence of the total error of the finite element discretisation for model problem A with Feng's conditions of order 0 to 5 in the mesh-width h for (a) $p = 2$ and (b) $p = 3$. The radius of the domain is $R = 8$.

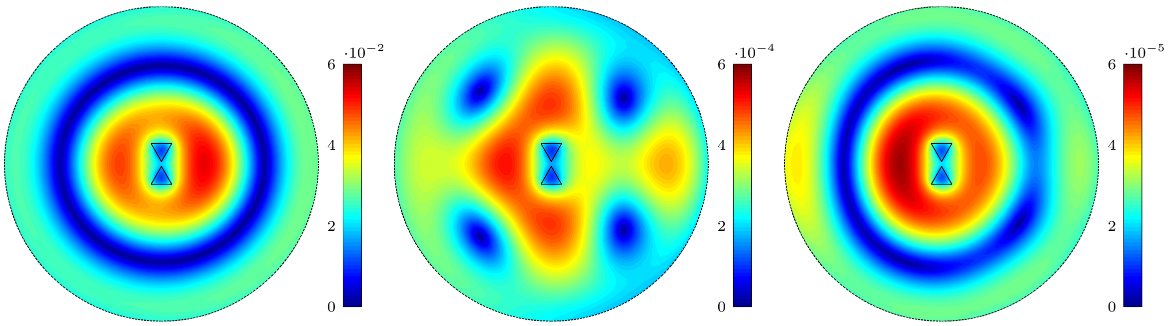

Figure 8. The error (absolute value) for model problem B with $R = 8$ on a fine mesh resolving the triangles and $p = 6$.

10 when using Feng-4 instead of Feng-2. For the parameters used the Feng-5 conditions do not give a further error reduction. This will only be achieved for larger domain radia R.

Total error for model problem A. Using the proposed finite element method for the scattering using Feng's conditions the discrete solution compromises a discretisation error and a modelling error. When fixing the domain, and let R large enough, then mesh refinement the error reduces due to a decrease of the discretisation error and saturates on the level of the modelling error. To obtain a certain level of the total error level the refinement of mesh might not be sufficient, where then either an higher order Feng condition has to be used or the radius of the domain is to be increased.

We have studied the total error for the model problem A with a fixed domain Ω with $R = 8$, and uniform polynomial degrees $p = 2, 3$. The total $L^2(\Omega)$-error as a function of the mesh-width h for Feng's conditions up to order 5 are shown in Fig. 7. Before reaching the level of the modelling error the total error decays like $O(h^3)$ or $O(h^4)$ for $p = 2$ or $p = 3$, respectively.

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